SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 180 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.

Total Marks – 120

- Attempt questions 1-10.
- All questions are of equal value.

Examiner: *D.McQuillan*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Total marks – 120
Attempt Questions 1–10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) How many degrees, to the nearest minute, are in 1 radian? **2**

(b) Rationalise the denominator of $\dfrac{2\sqrt{2}}{\sqrt{7}-\sqrt{3}}$. **2**

(c) Sketch a graph of $y=|2x-3|$. **2**

(d) Solve the inequality $2x^2+7x-15\geq 0$. **2**

(e) Evaluate $\sum_{k=0}^{19}(3k-1)$. **2**

(f) If $\log_e 5x-\log_e 2=2\log_e x$ find all real values of x. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\dfrac{dy}{dx}$ for the following

(i) $y = \tan(x^2)$ **2**

(ii) $y = 2x\sin(2x)$ **2**

(b)

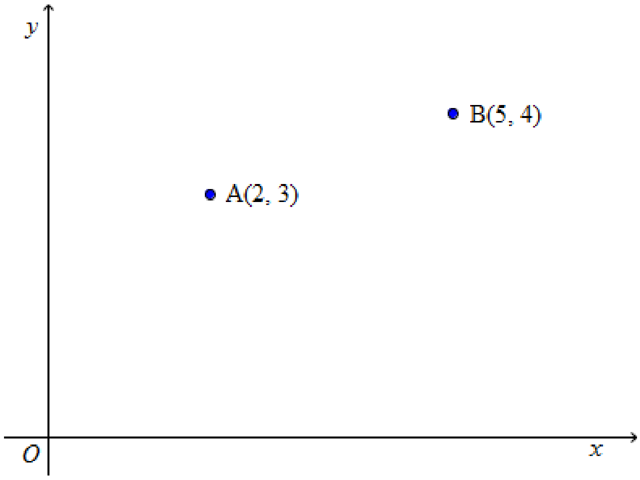
(i) Find $\displaystyle\int \frac{x^2}{x^3 - 1}\,dx$. **2**

(ii) Evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos\left(\frac{1}{2}x\right)dx$ in exact form. **3**

(c) Find the equation of the tangent to $y = \sin\left(x + \dfrac{\pi}{3}\right)$ at the point where $x = \pi$. **3**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows the points A(2, 3) and B(5, 4)



(i) Show that the equation of AB is $x - 3y + 7 = 0$. **2**

(ii) Find the coordinates of M, the midpoint of AB. **1**

(iii) Show that the equation of the perpendicular bisector of AB is $3x + y - 14 = 0$. **2**

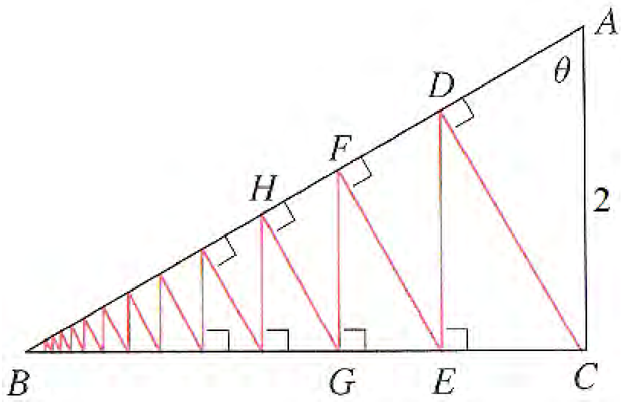
(iv) The perpendicular bisector of AB cuts the x-axis at C. Find the coordinates of C. **1**

(v) Find the area of triangle BCO. **2**

Question 3 continues on page 4

Question 3 (continued)

(b)



A right triangle ABC is given with $\angle A = \theta$ and $|AC| = 2$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, $EF \perp AB$, and this process is continued indefinitely as in the figure. Find the total length of all the perpendiculars

$|CD| + |DE| + |EF| + |FG| + \cdots$

in terms of θ. **4**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In Lower Warkworth the local doctor, based on years of data research, estimates that the probability of an adult catching influenza was 0.1 while the probability of a child catching the dreaded influenza was 0.3. The Blott family consists of Dad, Mum and two young Blotts. Calculate the probability that:

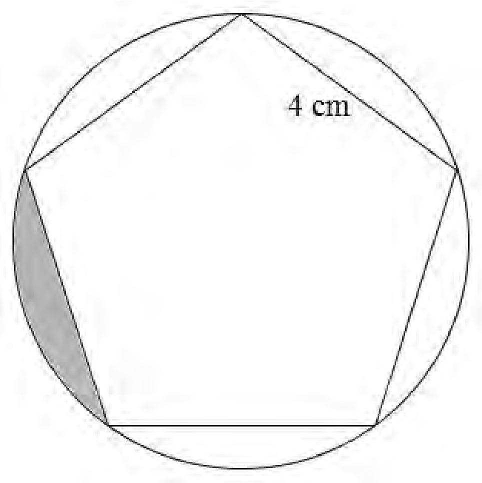
(i) both adults catch influenza **1**

(ii) only one child catches influenza **1**

(iii) exactly one adult and one child catches influenza **2**

(iv) at least one family member catches influenza. **2**

(b)



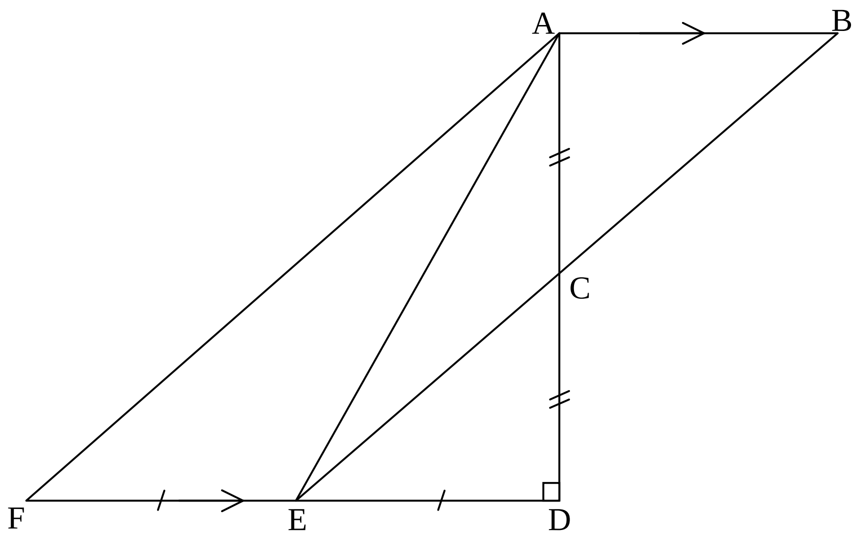
(i) Find an expression for the area of the regular pentagon with side length 4 cm. **3**

(ii) Find the radius of the circle to two decimal places. **2**

(iii) Hence or otherwise find the area of the shaded segment to two decimal places. **1**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram AB ∥ FD, ADF is a right-angled triangle, C is the midpoint of AD and E is the midpoint of FD.

(i) Explain why $\angle CED = \angle ABC$. **1**

(ii) Show that $\Delta CDE \equiv \Delta CAB$. **2**

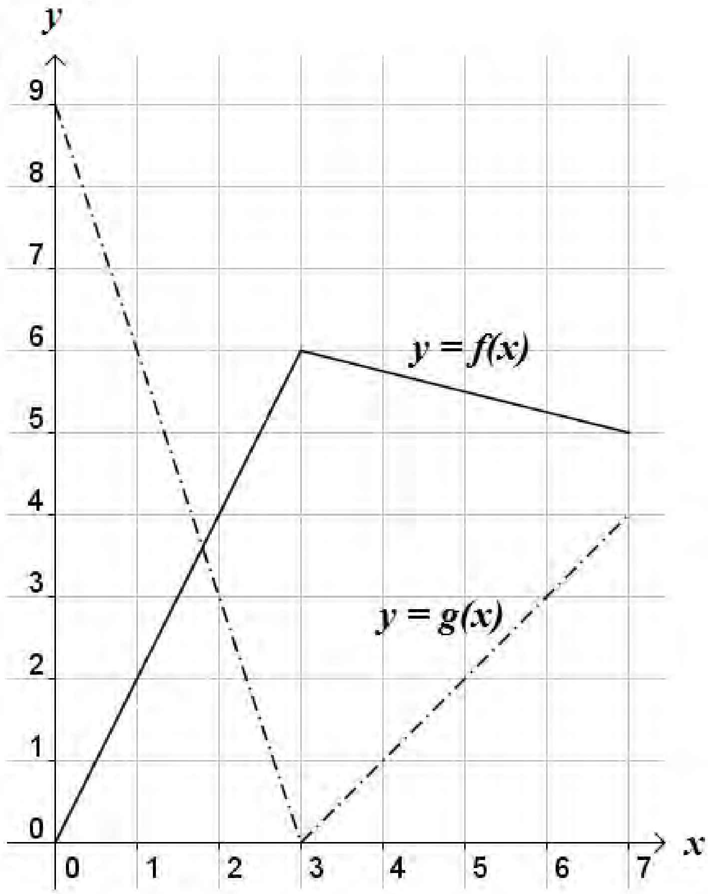
(iii) Show that $AF = 2BC$. **2**

(iv) Show that $\angle ACB = \angle DAF$. **1**

Question 5 continues on page 7

Question 5 (continued)

(b)



If $f(x)$ and $g(x)$ are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(g(x))$ find the value of

(i) $u'(1)$ **2**

(ii) $v'(1)$ **2**

(c) Show that if $|x+3| < \dfrac{1}{2}$, then $|4x+13| < 3$. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the curve $y = \dfrac{x}{x^2+1}$.

(i) Find the turning points and determine their nature. **3**

(ii) Find the points of inflection. **2**

(iii) Since x^2+1 is never zero the curve has no vertical asymptotes. Find the horizontal asymptotes by evaluating $\lim\limits_{x\to\infty} \dfrac{x}{x^2+1}$. **1**

(iv) Sketch the curve. **2**

(b) Tom is 60 years old and about to retire at the beginning of the year 2009. He joined a superannuation scheme at the beginning of 1969. He invested \$750 at the beginning of each year. Compound interest is paid at 9% per annum on the investment, calculate to the nearest dollar:

(i) The amount to which the 1969 investment will have grown by the beginning of 2009. **1**

(ii) The amount to which the total investment will have grown by the beginning of 2009. **3**

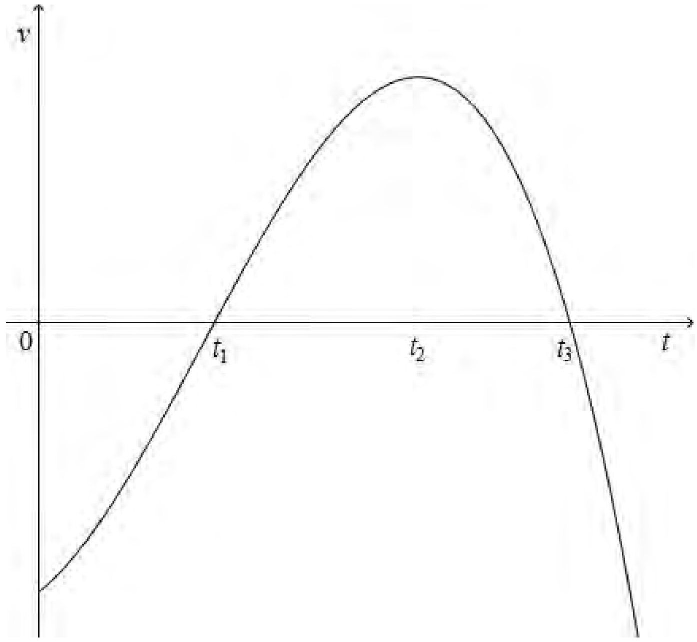
Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If α and β are the roots of the equation $3x^2 - 12x - 9 = 0$, find the values of:

(i) $\dfrac{1}{\alpha^3\beta^3}$ **1**

(ii) $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta}$ **2**

(b) A particle moves in a straight line and the graph shows the velocity *v* of the particle after time *t*.



(i) What is happening to the particle at t_1? **1**

(ii) What is happening to the particle at t_2? **1**

(iii) Sketch the graph of displacement *x*, as a function of *t*, if the particle is initially at the origin. **3**

(c) The locus of the point P(*x*, *y*) such that the sum of the squares of its distances from the points A(2, 4) and B(6, –8) is 118, is a circle. Find the centre and radius of the circle. **4**

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate $10^x + 10x$. **2**

(b) A particle moves in a straight line. At time *t* seconds its displacement *x* cm from a fixed point O on the straight line is given by:

$$x = t + \frac{1}{t+1}$$

(i) What is the initial displacement of the particle? **1**

(ii) When is the particle at rest? **2**

(iii) What is the acceleration after 5 seconds. **2**

(iv) What happens to the acceleration as *t* increases? What does this tell you about the velocity as *t* becomes large. **2**

(c) A petrol tank is designed by the rotation of the curve $y = \frac{1}{5}x(x-40)$ about the *x* axis between the planes $x = 0$, $x = 40$. If the units are in centimetres, how many litres would the tank hold? **3**

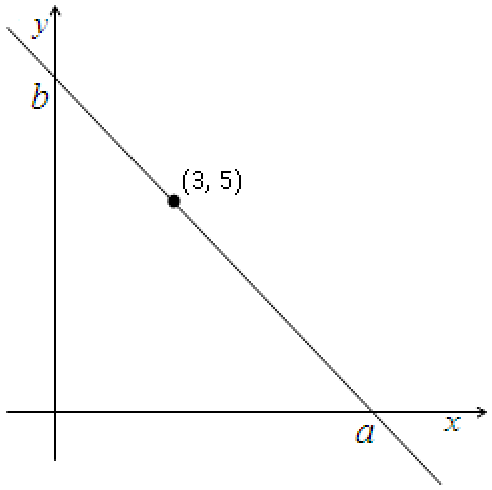
Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The population of a small town grows from 9000 to 11000 in 10 years.

(i) Find the annual growth rate to the nearest per cent, assuming it is proportional to the population. **2**

(ii) Calculate the population of the town 25 years after the initial count. **1**

(b)



(i) For the given figure show that $a = \dfrac{3b}{b-5}$. **2**

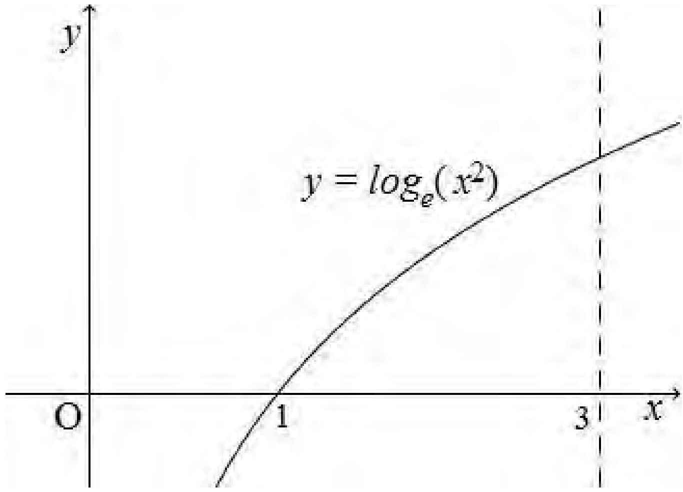
(ii) Find the equation of the line through the point (3, 5) that cuts off the least area from the first quadrant. **4**

(c) A ladder 2 metres long rests against a vertical wall. Let θ be the angle between top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \dfrac{\pi}{3}$. **3**

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $x\sin\pi x = \int_0^{x^2} f(t)dt$ find $f(4)$. **2**

(b) The graph of the function $y = \log_e(x^2)$ is shown below.



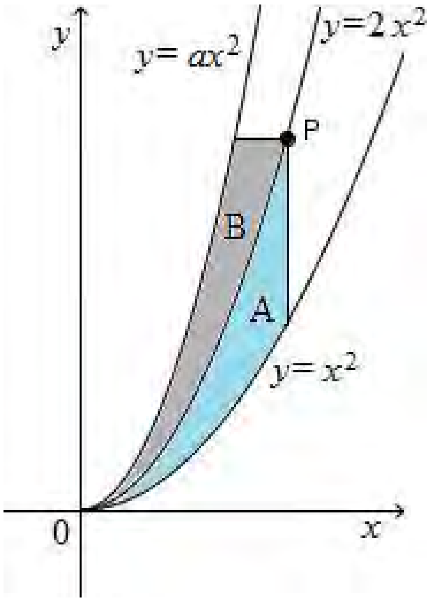
(i) Use the Trapezoidal rule with 5 function values to approximate $\int_1^3 \log_e\left(x^2\right)dx$ and explain why this approximation underestimates the value of the integral. **3**

(ii) Find $\int_0^{\ln 9} e^{\frac{y}{2}}dy$ and hence find the exact value of $\int_1^3 \log_e\left(x^2\right)dx$. **3**

Question 10 continues on page 13.

Question 10 (continued)

(c)



The figure shows a function $y = ax^2$ with the property that, for every point P on the middle function $y = 2x^2$, the area A and B are equal. Find the value of a. **4**

End of Paper

2U THSC 08

①

Question 1

(a) $1^c = \frac{180^\circ}{\pi}$

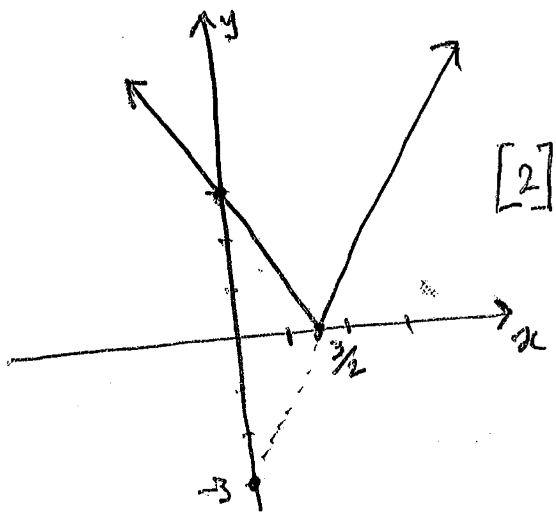
$\doteqdot 57^\circ 18'$ [2]

(b) $\frac{2\sqrt{2}}{\sqrt{7}-\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{7}-\sqrt{3}} \times \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

$= \frac{2\sqrt{14}+2\sqrt{6}}{7-3}$

$= \frac{1}{2}(\sqrt{14}+\sqrt{6})$ [2]

(c) $y = |2x-3|$

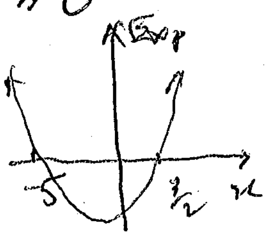


[2]

(d) $2x^2+7x-15 \geq 0$

$(2x-3)(x+5) \geq 0$

$x \leq -5, \ x \geq \frac{3}{2}$



[2]

(e) $\sum_{k=0}^{19} (3k-1) = -1+2+5+\dots+56$

This is an A.S.; $a=-1$, $l=56$, $n=20$

$S_{20} = \frac{20}{2}(-1+56)$

$= 550$ [2]

(f) $\ln 5x - \ln 2 = 2\ln x$

$\ln \frac{5x}{2} = \ln x^2$

$\therefore \frac{5x}{2} = x^2, \ x>0$

$2x^2-5x=0$

$x(2x-5)=0$

$x=0$ or $5/2$

But $x>0$

$\therefore x = 5/2$ [2]

Question 2

a) i $y = \tan(x^2)$

$\frac{dy}{dx} = 2x\sec^2(x^2)$ (1) (1)

ii $y = 2x\sin(2x)$

$\frac{dy}{dx} = 2x \times \cos(2x) \times 2 + \sin(2x) \times 2$

$= 4x\cos(2x) + 2\sin(2x)$ (1) (1)

b) i $\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3-1}$

$= \frac{1}{3}\log_e(x^3-1) + C$ (1) (1)

ii $\int_{\pi/2}^{\pi} \cos\left(\frac{1}{2}x\right)dx = \left[2\sin\left(\frac{1}{2}x\right)\right]_{\pi/2}^{\pi}$ (1)

$= 2\sin\frac{\pi}{2} - 2\sin\frac{\pi}{4}$ (1)

$= 2 - \sqrt{2}$ (1)

c) $y = \sin(x + \pi/3)$

$\frac{dy}{dx} = \cos\left(x + \frac{\pi}{3}\right)$

at $x = \pi$

$\frac{dy}{dx} = \cos\left(\frac{4\pi}{3}\right)$

$= -1/2$ (1)

when $x = \pi$

$y = \sin\left(\frac{4\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$ (1)

$\therefore$ pt $\left(\pi, -\frac{\sqrt{3}}{2}\right)$ $m = -1/2$

$y + \frac{\sqrt{3}}{2} = -\frac{1}{2}(x - \pi)$

$2y + \sqrt{3} = -1(x - \pi)$

$2y + \sqrt{3} = -x + \pi$

$x + 2y + \sqrt{3} - \pi = 0$ (1)

Question (3) [12 marks]

(a)

(i) $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{y-3}{x-2} = \frac{1}{3}$ (2)

$\therefore$ $\boxed{x - 3y + 7 = 0}$

(ii) $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

$\left(2 + 5/2, 3 + 4/2 \right)$

$\boxed{(7/2, 7/2)}$ (1)

(iii) $m_{AB} = 1/3$.

$\therefore$ grad. of perp. line to AB $= -3$.

$\therefore$ Equation of the perp. bisector of AB.

[ ]

$y - \frac{7}{2} = -3\left(x - \frac{7}{2}\right)$

$2y - 7 = -6x + 21$

$\therefore$ $\boxed{3x + y - 14 = 0}$

(2)

(iv) $3x + y - 14 = 0$

When $y = 0$, $x = 14/3$.

$\therefore$ $\boxed{C\left(14/3, 0\right) \quad 4\tfrac{2}{3}}$ (1)

(v)

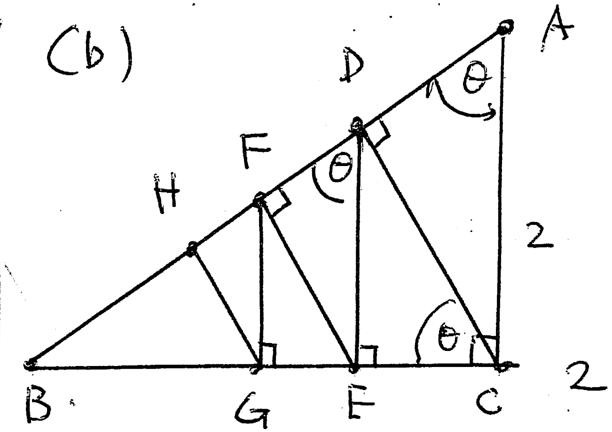
$A = \frac{1}{2} \times \frac{14}{3} \times 4\ 2.$

$= 28/3$

$= 9\frac{1}{3}$ (2)

$\boxed{\text{Area} = 9\tfrac{1}{3}}$

(b)



- In $\triangle ADC$, $\angle ACD = 90 - \theta$

$\therefore \angle DCE = \theta$.

$\frac{|DC|}{2} = \sin\theta \Rightarrow |DC| = 2\sin\theta$ —— (1)

- In $\triangle ECD$

$\frac{|DE|}{|DC|} = \sin\theta$

$\therefore |DE| = DC \sin\theta$ [From (1)

$= 2\sin^2\theta$ —— (2)

- In $\triangle DEF$ 2

$\frac{|FE|}{|DE|} = \sin\theta \Rightarrow \therefore FE = 2\sin^3\theta$

i.e sum

$2\sin\theta + 2\sin^2\theta + 2\sin^3\theta + \cdots$

$= 2(\sin\theta + \sin^2\theta + \cdots$

$= \boxed{\frac{2\sin\theta}{1 - \sin\theta}}$ (4)

Question (9) [12 marks]

(a) $\frac{dP}{dt} = kP$

(2) $P = P_0 e^{kt}$ ——— (1)

When $t = 0$, $P = 9000$

i.e. $P = P_0 = 9000$

When $t = 10$, $P = 11000$ 1

$\therefore 11000 = 9000 e^{10k}$

$\therefore e^{10k} = 11/9$ 1

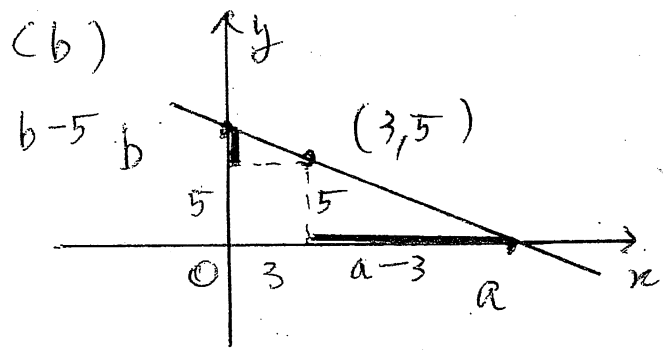
$$k = \frac{1}{10}\ln(11/9) = 0.02$$

$\therefore P = 9000 e^{25k}$

$= 9000 \times 1.65149$

(1) $$P = 14863$$

(b)



From similar Δ's

(i) $\frac{b-5}{3} = \frac{5}{a-3}$ 1

$\therefore a - 3 = \frac{15}{b-5}$

$\Rightarrow a = \frac{15}{b-5} + 3$

$= \frac{15 + 3b - 15}{b-5}$

(2) $$a = \frac{3b}{b-5}$$ 1

(ii) $A = \frac{1}{2}ab$

$\therefore A = \frac{3b^2}{2(b-5)}$ 1

$\frac{dA}{db} = \frac{12b(b-5) - 6b^2}{(b-5)^2}$

$0 = \frac{6b(b-10)}{(b-5)^2}$ 2

$\Rightarrow b = 0, 10$

When $b = 10$, $a = 6$.

Equation:

$y - 5 = -\frac{5}{3}(x - 3)$ 1

Equation

(4) $$5x + 3y - 30 = 0$$

Test

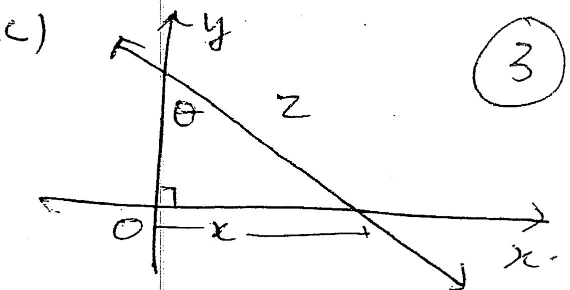
b	9	10	11
$\frac{dA}{db}$	$-27/8$ (−)	0	$11/6$ (+)

− ___/ +

$\therefore 5x + 3y - 30 = 0$

Cuts least Area.

(c) (3)



$\frac{x}{2} = \sin\theta$ 1

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$ 1

$$\left.\frac{dx}{d\theta}\right|_{\theta = \pi/3} = 2\cos\frac{\pi}{3} = 1$$

1 m/radian. 1

QUESTION 4

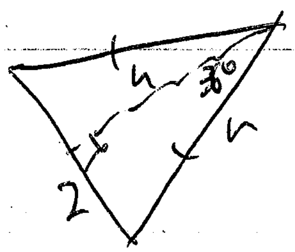
a (i) 0.01

(ii) $0.3\times0.7+0.7\times0.3=0.42$

(iii) $0.1\times0.9\times0.3\times0.7\times4=0.0756$

(iv) $1-0.9\times0.9\times0.7\times0.7=0.6031$

b (i)



$\tan 36° = \frac{2}{h}$

$h = \frac{2}{\tan 36°}$

$\text{Area} = 5\times\frac{1}{2}\times4\times h$

$= \frac{20}{\tan 36°}\ \text{cm}^2$

(ii) $\sin 36° = \frac{2}{r}$

$r = \frac{2}{\sin 36°}$

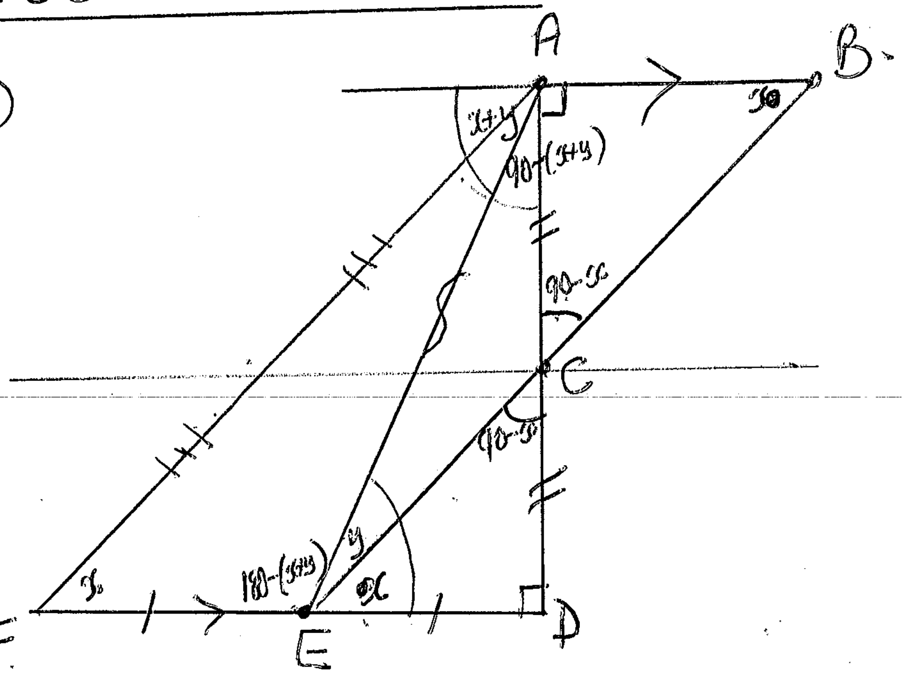
$= 3.40\ \text{cm}$

(iii) $\dfrac{\frac{4\pi}{\sin^2 36°} - \frac{20}{\tan 36°}}{5} = 1.77\ \text{cm}^2$

Trial HSC 2008 2 unit

12

(5) (a)



(i) alternate angles in // lines AB // FD, EB transversal (1)

(ii) show $\Delta CDE \equiv \Delta CAB$

$CD = AC$ given

$C\hat{E}D = A\hat{B}C$ alt. angles

$E\hat{C}D = A\hat{C}B$ vert. opp (2)

$\therefore \Delta CDE \equiv \Delta CAB$ AAS

Cannot use RHS, SSS, SAS.

(iii) show $AF = 2BC$

$\Delta AFE \equiv \Delta EBA$ (AAS).

ABEF is a parm

Line through midpt of 1 side of ΔADF // to a 3rd side FD, bisects the other side in proportion, so $AF = BE = BC + CD$, $EC = BC$ from (ii) (2)

$AF = 2BC$.

(iv) Show $A\hat{C}B = D\hat{A}F$

From diagram $A\hat{C}B = 90 - x$ (angle sum △)

$B\hat{A}C = 90^\circ$ // lines, transversal

$D\hat{A}F = D\hat{A}E + E\hat{A}F$

$= 90^\circ - (x+y) + y = 90^\circ - x$ (1)

b) $u(x) = f(x)g(x)$

$u'(x) = f(x)g'(x) + g(x)f'(x)$

given $y = f(x)$ slope $= 2$ $\quad 0 \le x \le 3$

$= -\frac{1}{4}$ $\quad 3 \le x \le 7$

given $y = g(x)$ slope $= -3$ $\quad 0 \le x \le 3$

$= 1$ $\quad 3 \le x \le 7$

i) $u'(1) = f(1)g'(1) + g(1)f'(1)$

$= 2 \times -3 + 6 \times 2$ (2)

$= 6$

ii) $V(x) = f(g(x))$

$V'(x) = f'(g(x)) \times g'(x)$

so $V'(1) = f'(g(1)) \times g'(1) = f'(6) \times g'(1) = -\frac{1}{4} \times -3 = \frac{3}{4}$ (2)

c) $|4x+13| < 3$

$|4x+13| \le |4x+12| + |1|$

$= 4|x+3| + 1$

$< 4 \times \frac{1}{2} + 1$

$= 3$

or using $|x+3| < \frac{1}{2}$

$-\frac{1}{2} < x+3 < \frac{1}{2}$

$\times 4$ $\quad -2 < 4x+12 < 2$

$+1$ $\quad -1 < 4x+13 < 3$

So $-3 < 4x+13 < 3$ is also true

$\therefore |4x+13| < 3$

(2)

QUESTION 6 2U Trial 2008

$y = \dfrac{x}{x^2+1}$ Let $u = x$ $u' = 1$
$v = x^2+1$ $v' = 2x$.

$\dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2} = \dfrac{x^2+1-2x}{(x^2+1)^2}$.

Turning points when $\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{1-x^2}{(x^2+1)} = 0$.

$x = \pm 1$.

Turning points at $(+1, \frac{1}{2})(-1, \frac{-1}{2})$

NATURE

x	-2	-1	0	1	2
dy/dx	$-3/25$	0	1	0	$-\frac{3}{25}$
gradient	$\backslash$	—	/	—	$\backslash$
		Min		max	

$\therefore$ min at $(-1, -\frac{1}{2})$
max at $(1, \frac{1}{2})$.

b) Points of inflexion occur when $d^2y/dx^2 = 0$.

$\dfrac{d^2y}{dx^2} = \dfrac{vu' - uv'}{v^2}$ $u = 1-x^2$
$u' = -2x$.
$v = (x^2+1)$
$v' = 4x(x^2+1)$

$\dfrac{d^2y}{dx^2} = \dfrac{(x^2+1)^2(-2x) - (1-x^2)4x(x^2+1)}{(x^2+1)^4}$.

$= \dfrac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3}$.

$\dfrac{d^2y}{dx^2} = \dfrac{2x^3 - 6x}{(x^2+1)^3} = 0$.

$\therefore$ $2x(x^2-3) = 0$.
$x = 0$ $x = \pm\sqrt{3}$.

Points of inflexion are.

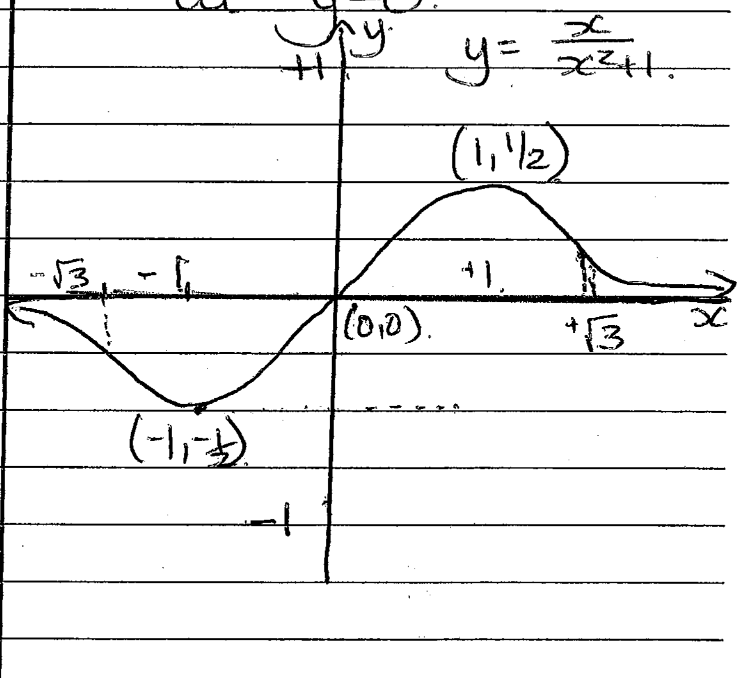
$(0,0), (-\sqrt{3}, \frac{-\sqrt{3}}{4}), (+\sqrt{3}, \frac{+\sqrt{3}}{4})$.

c) $\lim\limits_{x\to\infty} \left(\dfrac{x}{x^2+1}\right) = \lim\limits_{x\to\infty} \left(\dfrac{x/x^2}{x^2/x^2 + \frac{1}{x^2}}\right)$

since $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

$\lim\limits_{x\to\infty} \dfrac{x}{x^2+1} = 0$.

Horizontal assymptote at $y = 0$.



Q6 b)

i) $A = 750(1\cdot 09)^{40}$

$= 23557\cdot 065$

$= \$23557$

nearest \$.

ii) Yr2 $750(1\cdot 09)^{40} + 750(1\cdot 09)^{39}$

Yr3. $750(1\cdot 09)^{40} + 750(1\cdot 09)^{39} + 750(1\cdot 09)^{38}$.

Y40. $750(1\cdot 09^{40} + 1\cdot 09^{39} + \dots + 1\cdot 09)$

S_{40} when $a = 1\cdot 09$ $n = 40$ $r = 1\cdot 09$.

$S_n = \dfrac{a(r^n - 1)}{r-1}$

$S_{40} = \dfrac{1\cdot 09(1\cdot 09^{40} - 1)}{0\cdot 09}$.

$Y_{40} = 750 \times \dfrac{1\cdot 09(1\cdot 09^{40} - 1)}{0\cdot 09}$

$= 276218.898$

$= \$276219$ nearest \$.

Q7 (a) given $3x^2 - 12x - 9 = 0$ with roots α, β.

$$\alpha + \beta = -\frac{b}{a} = \frac{12}{3} = 4$$

$$\alpha\beta = \frac{c}{a} = \frac{-9}{3} = -3$$

(i) $\frac{1}{\alpha^3\beta^3} = \frac{1}{(\alpha\beta)^3} = \frac{1}{(-3)^3} = \frac{-1}{27}$ ①

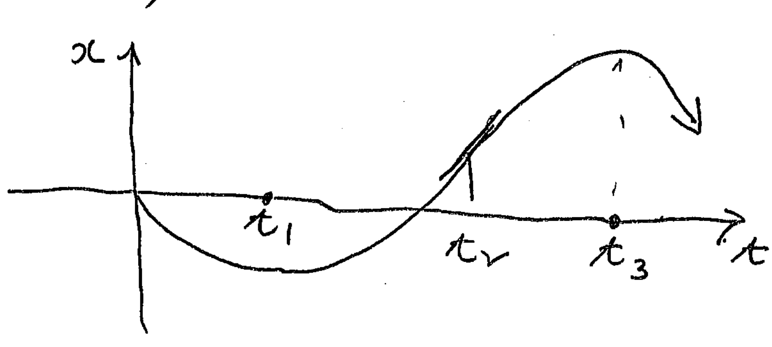
(ii) $\frac{\beta}{\alpha} + \frac{\alpha}{\beta} = \frac{\beta^2 + \alpha^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{16 + 6}{-3}$

$= \frac{-22}{3}$ ②

(b) (i) The particle is moving to the left, then stops at t_1 then moves to the right ①

(ii) The particle reaches its maximum velocity at t_2, then starts to slow down. ①

(iii)



③

(c) $AP^2 + BP^2 = 118 \Rightarrow (x-2)^2 + (y-4)^2 + (x-6)^2 + (y+8)^2 = 118$ ④

$\therefore x^2 - 4x + 4 + y^2 - 8y + 16 + x^2 - 12x + 36 + y^2 + 16y + 64 = 118$

$2x^2 + 2y^2 - 16x + 8y + 120 = 118$

$x^2 + y^2 - 8x + 4y = -1$

$(x-4)^2 + (y+2)^2 = -1 + 16 + 4$

$= 19$

$\therefore$ centre $(4, -2)$ radius $\sqrt{19}$.

2U THSC 08

Question 8

(a) $y = 10^x + 10x$

$\frac{dy}{dx} = 10^x \ln 10 + 10$ [2]

(b) $x = t + \frac{1}{t+1}$

(i) When $t=0$, $x = 1$ [1]

(ii) $\dot{x} = 1 - \frac{1}{(t+1)^2}$

$\dot{x} = 0$ when

$1 - \frac{1}{(t+1)^2} = 0$ $\quad t \neq -1$

$(t+1)^2 - 1 = 0$

$t^2 + 2t + 1 - 1 = 0$

$t^2 + 2t = 0$

$t(t+2) = 0$

$t = 0, -2$

(-2 is extraneous)

$\therefore t = 0$

Particle is initially at rest. [2]

(iii) $\ddot{x} = \frac{2}{(t+1)^3}$

$\ddot{x}(5) = \frac{2}{6^3}$

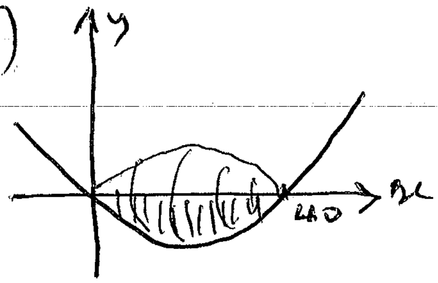
$= \frac{1}{108}$ cm/sec²

$\doteqdot 9.2593 \times 10^{-3}$ cm/s² [2]

(iv) $\ddot{x} \to 0$ as $t \to \infty$,

so $\dot{x} \to$ a limit of 1 cm/sec. [2]

(c)



$V = \pi \int_0^{40} \left[\frac{1}{5}(x^2 - 40x)\right]^2 dx$

$= \frac{\pi}{25} \int_0^{40} (x^4 - 80x^3 + 1600x^2)\, dx$

$= \frac{\pi}{25} \left[\frac{x^5}{5} - \frac{80x^4}{4} + \frac{1600x^3}{3} \right]_0^{40}$

$= \frac{\pi}{25} (341333\tfrac{1}{3})$

$\doteqdot 428932.117$ cm³

$\doteqdot 429$ L [3]

$2x \quad -3$

$x \quad 5$

t^2+2t-1

$\frac{d}{dx}(t+1)^{-1}$

$t \quad 1$

$t \quad -1$

$= \frac{-1}{(t+1)^2}$

$\dot{x} = 1-(t+1)^{-2}$

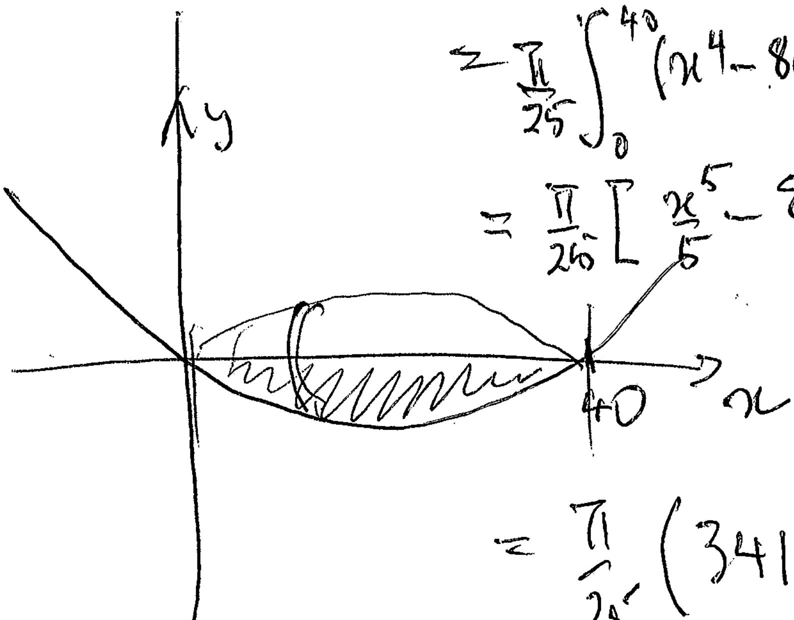
$\ddot{x} = -(-2)(t+1)^{-3}$

$= \frac{2}{(t+1)^3}$

$V = \pi \int_0^{40} \left[\frac{1}{5}(x^2-40x)\right]^2 dx$

$= \frac{\pi}{25}\int_0^{40} (x^4-80x^3+1600x^2)\, dx$

$= \frac{\pi}{25}\left[\frac{x^5}{5}-\frac{80x^4}{4}+\frac{1600x^3}{3}\right]_0^{40}$



$= \frac{\pi}{25}\left(3413333\frac{1}{3}\right)$

$\doteqdot 428932.117 \text{ cm}^3$

$\doteqdot 429 \text{ L}$

2008 Trial HSC Mathematics:
Solutions— Question 10

10. (a) If $x \sin \pi x = \int_0^{x^2} f(t)\,dt$, find $f(4)$. [2]

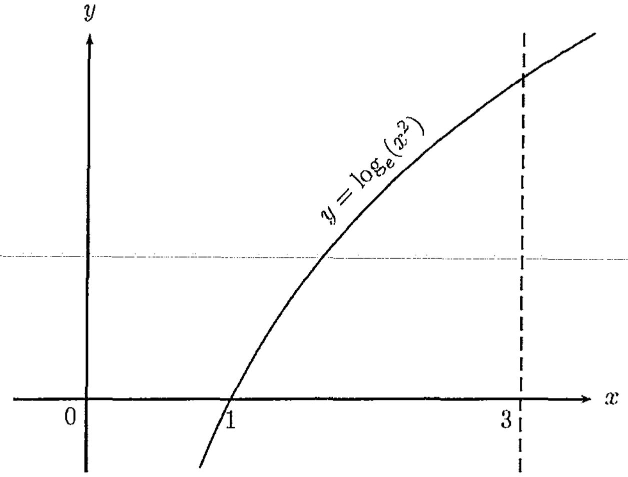
Solution: Let $x^2 = u$,

$$\begin{aligned}
f(u) &= \frac{d}{du}\int_0^u f(t)\,dt,\\
&= \frac{d}{du}\{\pm\sqrt{u}\sin(\pm\pi\sqrt{u})\},\\
&= \pm\sqrt{u}\cos(\pm\pi\sqrt{u}) \times \frac{\pi}{\pm 2\sqrt{u}} + \frac{\sin(\pm\pi\sqrt{u})}{\pm 2\sqrt{u}}.\\
\therefore\ f(4) &= \pm\sqrt{4}\cos(\pm\pi\sqrt{4}) \times \frac{\pi}{\pm 2\sqrt{4}} + \frac{\sin(\pm\pi\sqrt{4})}{\pm 2\sqrt{4}}.\\
&= \frac{2\pi}{4} + 0,\\
&= \frac{\pi}{2}.
\end{aligned}$$

Solution: Alternative method

$$\begin{aligned}
F(x^2) - F(0) &= x\sin(\pi x),\\
2xF'(x^2) - 0F'(0) &= \sin(\pi x) + \pi x\cos(\pi x),\\
F'(x^2) &= \frac{\sin(\pi x) + \pi x\cos(\pi x)}{2x},\\
&= f(x^2).\\
\text{When } x^2 &= 4,\\
x &= \pm 2.\\
\therefore\ f(4) &= \frac{\sin(\pm 2\pi) \pm 2\pi\cos(\pm 2\pi)}{\pm 4},\\
&= \frac{0 + 2\pi}{4},\\
&= \frac{\pi}{2}.
\end{aligned}$$

(b) The graph of the function $y = \log_e(x^2)$ is shown below.

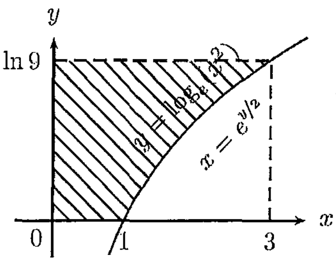


(i) Use the Trapezoidal rule with 5 function values to approximate $\int_1^3 \log_e(x^2)\,dx$ and explain why this approximation underestimates the value of the integral. **3**

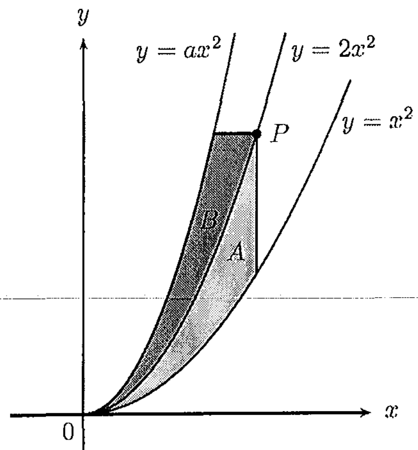
Solution: $\int_1^3 \ln(x^2) \approx \frac{0.5}{2}\{0 + 2(0.8109 + 1.3863 + 1.8326) + 2.1972\}$,
≈ 2.564 (3 sig. fig.).
Each trapezium's sloping edge is under the curve as the curve is always concave downwards. The approximation is short by the amounts between the top of the trapezia and the curve.

(ii) Find $\int_0^{\ln 9} e^{\frac{y}{2}}\,dy$ and hence find the exact value of $\int_1^3 \log_e(x^2)\,dx$. **3**

Solution:
$$\int_0^{\ln 9} e^{\frac{y}{2}}\,dy = \left[2e^{\frac{y}{2}}\right]_0^{2\ln 3},$$
$$= 2e^{\frac{2\ln 3}{2}} - 2 \times 1,$$
$$= 6 - 2,$$
$$= 4.$$
$$\therefore \int_1^3 \ln(x^2)\,dx = 3\ln 9 - 4.$$



(c) $\boxed{4}$



The figure shows a function $y = ax^2$ with the property that, for every point P on the middle function $y = 2x^2$, the areas A and B are equal.
Find the value of a.

Solution:

$$
\begin{aligned}
\text{Let } P &= (p,\, q). \\
\text{Area } A &= \int_0^p \left(2x^2 - x^2\right) dx, \\
&= \int_0^p \left(x^2\right) dx, \\
&= \left[\frac{x^3}{3}\right]_0^p, \\
&= \frac{p^3}{3}. \\
\text{Area } B &= \int_0^q \left(\sqrt{\frac{y}{2}} - \sqrt{\frac{y}{a}}\right) dy, \\
&= \left[\frac{2y^{3/2}}{3\sqrt{2}} - \frac{2y^{3/2}}{3\sqrt{a}}\right]_0^q, \\
&= \frac{2}{3}q^{3/2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{a}}\right). \\
\text{But } q &= 2p^2, \\
\text{so area } B &= \frac{2}{3}.2^{2/3}.p^3\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{a}}\right). \\
\text{Now } A &= B, \\
\textit{i.e. } \frac{p^3}{3} &= \frac{4p^3}{3}\left(1 - \sqrt{\frac{2}{a}}\right), \\
1 &= 4 - 4\sqrt{\frac{2}{a}}, \\
\sqrt{\frac{2}{a}} &= \frac{3}{4}, \\
\frac{2}{a} &= \frac{9}{16}, \\
a &= \frac{32}{9}.
\end{aligned}
$$